

Graph Theory Problem Set 3

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1. Show that if G is a graph with n vertices and at least $n^2/4 + 1$ edges, then G contains at least $\lfloor n/2 \rfloor$ triangles.

Trivial for $n < 4$. Let $n \geq 4$. A triangle is formed when for neighbors $u, v \in V(G) \exists x$ s.t. $x \in N(u), x \in N(v)$. WLOG let $d(u) = \Delta(G)$. Combining $\Delta(G) \geq \bar{d}$, $2e(G)/v(G) = \bar{d}$ and $e(G) > n^2/4$, we observe $\Delta(G) > n/2$.

We will now try to find an upper bound to $\Delta(G)$. Try $\Delta(G) = n - 1$, then $d(v) = 1$ for all $v \in N(u)$. This contains $n - 1 < n^2/4 + 1$ edges. A similar construction for maximum edges with $\Delta(G) = n - 2$ while minimizing triangles is trivial to derive and yields $2n - 4 < n^2/4 + 1$ edges. Again constructing for maximum edges with $\Delta(G) = n - 3$ while still minimizing triangles, we begin to see the structure of the complete bipartite graph emerge. By constructing a $K_{n/2, n/2}$ we can maximize the edges of G while still minimizing triangles, better yet there are zero of them. Note this construction is forbidden by $\Delta(G) > n/2$, however any other construction would have less edges, for odd n the construction is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Observe $e(K_{n/2, n/2}) = n^2/4$ thus we still must add another edge. The addition of any edge to $K_{n/2, n/2}$ would form a triangle and immediately satisfy $\Delta(G) > n/2$, however by Mantel's Theorem there is no way to add this edge without creating a triangle. Observe the addition of an edge to either side a of a $K_{a, b}$ or side b , adds b or a triangles respectively. Thus the amount of triangles is at least $\lfloor n/2 \rfloor$. \square

2. Show that for all $\varepsilon > 0$, there is some $\delta > 0$ such that the following holds: if G is a graph with n vertices and at least $(1/4 + \varepsilon)n^2$ edges, then G contains at least δn^3 triangles.

Let $\delta = 1/n^3$, and this result is trivial as it follows from Q.1 and Mantel's Theorem.

Let $\delta = f(\varepsilon)$, then with a result, $t(G) \geq (e(G)/3n)(4e(G) - n^2)$ for $e(G) > n^2/4$, by E. A. Nordhaus and B. M. Stewart in their paper "Triangles in an ordinary graph" (1963) we can find: $((1/4 + \varepsilon)n^2/3n)((1 + 4\varepsilon)n^2 - n^2) = 4\varepsilon n^2(1/4 + \varepsilon)(n/3) = (\varepsilon(1 + 4\varepsilon)/3)n^3 = n^3(\varepsilon + 4\varepsilon^2)/3$. Where $t(G)$ is the number of triangles in G . Thus $\delta = (\varepsilon + 4\varepsilon^2)/3$. \square

3. Show that in any red-blue coloring of the edges of K_n , there are at least $(1/4 - o(1))\binom{n}{3}$ monochromatic triangles.

$n \geq 3$. Note $\binom{n}{3}$ triangles in a K_n . Working backwards from a full red K_n we observe that by just recoloring the external edges for $n = 4, 5$ eliminates all monochromatic triangles. More generally it will remove $(n-2) + (n-2)(n-3) + (n-4)$ monochromatic triangles which equals $\binom{n}{3}$ for $n = 4, 5$. Note that for $n = 6$, this recoloring process would remove 18 out of 20 but any other edge recoloring would form a blue triangle. This is where we now observe that the roles of red and blue edges or triangles are symmetric. This allows us to fix a coloring, choose a random isomorphism of the K_n , a random labeling of the vertex set and find the amount of monochromatic triangles in expectation. It is trivial to see that the probability a triangle is monochromatic as $n \rightarrow \infty$ is $1/4$ (coloring as now been randomized and due to symmetry $p = 1/2$ will yield best results) and lower for small n as shown before. Thus we arrive at there are at least $(1/4 - o(1))\binom{n}{3}$ monochromatic triangles. \square

4. Using dependent random choice or otherwise, show that if G is a graph with n vertices and εn^2 edges, then G contains a 1-subdivision of a complete graph on $\Omega(\varepsilon^{3/2}n^{1/2})$ vertices.

Note $\varepsilon < 1/2$. Let $\varepsilon^{3/2}n^{1/2} = n'$. Let H denote the 1-subdivision of a complete graph on n' vertices. Observe H is a subgraph of $K_{n', \binom{n'}{2}}$, $e(H) = 2\binom{n'}{2}$, $v(H) = n' + \binom{n'}{2}$, $\bar{d}(G) = 2\varepsilon n$.

DRC: Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with $|V| = n$ vertices and average degree $\bar{d}(G) = 2|E(G)|/n$. If there is a positive integer t such that:

$$\frac{\bar{d}(G)^t}{n^{t-1}} - \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

We wish to find a set of n' vertices such that every pair of vertices has a distinct common neighbor that we can use for the subdivision of $K_{n'}$. In terms of DRC we want $r = 2$, $a = n'$, and we hope to find a t s.t $m = \binom{n'}{2} = (\varepsilon^3 n)/2 - (\varepsilon^{3/2}n^{1/2})/2$ therefore we have room to start assigning a system of distinct representatives.

$$\frac{(2\varepsilon n)^t}{n^{t-1}} - \binom{n}{2} \left(\frac{(\varepsilon^3 n)/2 - (\varepsilon^{3/2}n^{1/2})/2}{n}\right)^t \geq \varepsilon^{3/2}n^{1/2}$$

$$\frac{(2\varepsilon n)^t}{n^{t-1}} - \frac{n(n-1)}{2} \left(\frac{(\varepsilon^3 n)/2 - (\varepsilon^{3/2} n^{1/2})/2}{n} \right)^t \geq \varepsilon^{3/2} n^{1/2}$$

This turns out to be unsolvable for t with general $0 < \varepsilon < 1/2$ and $n \in \mathbb{N}$, implying my choice of m is too high, however despite my best efforts I cannot figure out how to find a bound on m . I believe my choice of m to be the minimum needed to greedily assign a SDR, implying there is also something I do not understand about potential construction processes of H . Perhaps there is some asymptotics trick I do not know to derive an upper bound on m , this is my first class working with asymptotics. I also have found numerous seemingly conflicting definitions of DRC online, which have me wondering if the equations or my interruptions are even correct. Neither textbook was helpful and I'm truly stumped. I believe the upper bound on m resides somewhere between $c\varepsilon^2 n$ and $c\varepsilon^3 n$ which allows the assignment. I also note it is possible to solve for t above with some given values of ε and n and not others, implying if my equation is correct there is some unstated relationship $f(n) \geq g(\varepsilon)$ that is also required to hold. ■

5. The dependent random choice lemma we proved shows that every sufficiently dense graph on n vertices contains a large set of vertices U with the useful property that every small subset of U has many common neighbors. Show that it is impossible to guarantee that both the size of U and the number of common neighbors are linear in n : by considering the graph on $\{0, 1\}^k$ where vertices are joined if their Hamming distance is at most $k/2$ (or otherwise), show that for infinitely many $n \in \mathbb{N}$, there is a graph G on n vertices with at least $n(n-2)/4$ edges such that any subset of G of linear size contains a pair of vertices with at most $o(n)$ common neighbors.

Note $n = 2^k$, as $k \in \mathbb{N}$ there are infinitely many graphs on $n \in \mathbb{N}$ with the form $\{0, 1\}^k$ where vertices are joined if their Hamming distance is at most $k/2$. For strings (or vertices) $x, y \in \{0, 1\}^k$, let $H(x, y)$ denote their Hamming distance. $E(e(G)) = P(H(x, y) \leq k/2) \binom{n}{2}$. Observe if we allow $x = y$, the R.V. $H(x, y)$ takes on the range $[0, k]$ and is binomial and symmetric around $k/2$. Thus, $E(e(G)) = \binom{n}{2}/2 = n(n-1)/4 > n(n-2)/4$. ✓

Let z be a common neighbor of x & y and $H(x, y) \leq k/2$. z can differ roughly in half the common digits in x & y and half the uncommon digits of x in y . Note if z differs from an uncommon digit of x in y than it agrees in y , by binarity and the roles of x and y are symmetric. This yields the result that the size of the common neighborhood of x and y is bounded above by $(\sum_{i=0}^{k/4} \binom{k/2}{i})^2 \ll 2^k$ thus $o(n)$. □

6. A subset S of $\{1, \dots, n\}$ is sum-free if there are no solutions to $x+y = z$ with $x, y, z \in S$. Show that the set $\{1, \dots, n\}$ has at most $2^{n/2+o(n)}$ sum-free subsets.

Observe the set of all odd numbers, $\{1, 3, \dots, n\}$ for odd n or $\{1, 3, \dots, n-1\}$

for even n , is sum-free. Trivially then any subset of the set of all odd numbers is also sum-free. Thus we have $2^{n/2}$ for even n and $2^{(n+1)/2}$ for odd n . Another set $\{\lfloor n/2 \rfloor + 1, \dots, n\}$ results in a similar amount of $2^{\lceil n/2 \rceil}$ subsets. There exist other "manufactured" sum-free subsets, even those composed of only small even numbers, such as $\{2, 6, 10\}$, however these sets are increasing sparse as n increases due to the exclusionary nature of the sum-free condition. For example if we include 8 in our subset, we eliminate its double 16, as well as all numbers of the form $8 + y$ where y is another element of our set, meaning for large n , the inclusion of one more number to a set S with $|S| = x$ would exclude x other numbers from that set. Thus these "manufactured" sets of S are bounded in size by roughly $n/2$. Therefore we have the result that the set $\{1, \dots, n\}$ has at most $2^{n/2+o(n)}$ sum-free subsets. \square