Graph Theory Problem Set 2

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1. Show that it is possible to orient the edges of any graph G in such a way that the in-degree and the out-degree of each vertex differ by at most 1.

Construct an arbitrary orientation of the edges of G.
We may assume that this orientation does not satisfy the statement, thus $\exists x \in V(G)$ s.t. $|d_{in}(x) - d_{out}(x)| > 1$.
WLOG say $d_{in}(x) > d_{out}(x)$ and $|d_{in}(x) - d_{out}(x)| = k$.

Reverse the orientation of an edge into x so it now flows from x into $y_i \in N(x)$ and do this $k-1$ times. Suppose y_i is now also problematic. Reverse a different edge that flows into y_i then the one you already reversed that connects y_i to some $z_i \in N(y_i)$. Repeating the process for z_i if it also becomes problematic, we begin to form a directed path along the vertices of G that begins at y_i.

Since G is finite, this process must terminate, either we will eventually form a cycle or find a vertex with only one edge. If we find a vertex with only one edge then the statement now holds for all vertices along the path, if we find a cycle that reconnects into our path at some vertex c that could be y_i, z_i or any other vertex along this directed path, then this cycle contributes zero to the sum of $|d_{in}(c) - d_{out}(c)|$ and can be ignored. Restart the process starting from c until you find the problematic terminating directed path then iterate back to the next problematic y_i. This path must exist, as if all other possible problematic paths out of c eventually form cycles, they would all contribute zero to $|d_{in}(c) - d_{out}(c)|$ and c wouldn't be problematic.

All y_i's and the vertices that failed the statement in the cascade are now no longer problematic and neither is x, repeating this process over all problematic x satisfies the statement.
$\square$

2. Let G be a graph on n vertices in which the largest independent set has size r. Show that G contains a path on at least n/r vertices.

Begin by forming a collection of independent sets of G in this manner:
Form C_1 by taking the largest independent set of $V(G)$.
Form C_2 by taking the largest independent set of $V(G) \setminus C_1$.
Form C_3 by taking the largest independent set of $V(G) \setminus \bigcup_{i=1}^{2} C_i$.
.
.
.
Form C_k by taking the largest independent set of $V(G) \setminus \bigcup_{i=1}^{k-1} C_i$.

Observe $\forall x \in C_i$, $1 < i \leq k$, that x must have an edge with a vertex in C_{i-1}, otherwise x could be placed in C_{i-1} and make it larger.
Look at any vertex in C_k find it's neighbor in C_{k-1}, similarly find that vertices neighbor in C_{k-2}, continuing all the way to C_1.
You have now formed a path on k vertices.
Since $|C_1| \geq |C_i|\ \forall i$, $k \geq n/|C_1| = n/r$.
□

3. Show that the chromatic number of any graph G equals one plus the length of the longest directed path in an orientation of G chosen to minimize this path's length.

Let $\chi(G) = h$. As I showed in question 7, any graph with $\chi(G) = h$, contains a complete K_h subgraph.
I will show by induction that any K_h contains a longest directed path in an orientation chosen to minimize this path's length of length $h-1$.
Trivial for $h = 1, 2$.
Let the statement hold for $h-1$ vertices. Consider the addition of the h-th vertex. Since we are inducting over complete graphs, the h-th vertex must connect to every other. Let P be our longest path in K_{h-1}, length of $P = h-2$, notice if you took the first vertex in P and reversed the edge to the next vertex $y \in P$, the length of P would decrease by one, but since our orientation was chosen to minimize P, the first vertex must have a path of length $h-3$ that flowed out of it from the opposite direction of P and would form a path of length $h-2$ beginning at y. Similarly no edge direction swap along P decreases the length of the longest path in K_{h-1}.

Let Q be our longest path in K_h, obviously length $Q \geq$ length $P = h-2$. To minimize Q we would want to attached the h-th vertex s.t it flows away from the first vertex in P and towards the last vertex in P, otherwise we would form an extension of P. Consider orientating the edge joining the second to last vertex of P, call it x, to the h-th vertex, if you allow $x \to h$, then you form a longer path by inserting h between x and the last vertex in P, thus $x \leftarrow h$.

Repeating the same logic for the third to last vertex, say v, shows if you allow $v \to h$ you can insert h between x and v and form a longer path, thus $v \leftarrow h$. This process terminates however when we reach the first vertex in P, which we had to allow to flow into h, thus length of $Q = h - 1$.

Since $K_h \subseteq G$, the length of the longest path in G under an orientation designed to minimize this value is $\geq h - 1 = \chi(G) - 1$.
Consider now beginning with an undirected graph and coloring it with a numbered set of colors, we will direct the edges s.t. they always flow from the lower numbered color to the higher colored one, clearly there can exist a directed path along the colors that has length $h - 1$, thus the minimum length directed path is bounded above by $h-1$ but also bounded below by $h-1$, thus they are equal.
$\square$

4. A kernel of a digraph D is an independent set $I \subset V(D)$ such that every vertex $v \in V(D) \setminus I$ can be reached from some vertex in I in one (directed) step. Show that there are directed graphs that do not have a kernel. A quasi-kernel of a digraph D is an independent set Q such that every vertex can be reached in at most two (directed) steps from Q. Show that every directed graph has a quasi-kernel.

A directed graph with no kernel:
Imagine a directed cycle on 3 vertices, x, y, z with edges x to y, y to z, z to x.
Claim:
No independent set can form a kernel of this graph.
Proof of Claim:
No set of ≥ 2 vertices is independent. Any set of one vertex will have a vertex two directed steps away. ✓

Let G be a graph on n vertices. Let $n \leq 3$.
Obviously G has a quasi-kernel.
Assume the property holds for $v(G) = n$ and consider $v(G) = n + 1$.
Pick $v \in V(G)$ s.t. $d_{out}(v) = \Delta_{out}(G)$, i.e v is a vertex with maximal out degree.
Let $X \subset V(G) = \{x : v \to x\}$.
Let G' be an induced subgraph on the vertex set $V(G) \setminus (X \cup \{v\})$.
By induction $\exists\ Q'$ s.t. Q' is a quasi kernel of G'.
If $Q' \to v$, $Q = Q'$, done. ✓
If $Q' \not\to v$, since $\forall x\ x \in X$, $x \notin Q'$, v is independent of Q', $Q = \{v\} \cup Q'$. ✓
Thus $\forall y \in V(G) \setminus (X \cup \{v\})$, y reachable ≤ 2 steps from Q.
$\forall x \in X$, x reachable in one or two step(s) from Q depending if $v \in Q$.
Finally, v reachable in zero or one step(s) from Q depending if $v \in Q$.
$\square$

5. Let $k \geq 2$. Prove that if G is k-vertex-connected, then every set of k vertices is contained in a cycle. Is the converse true?

Let $k = 2$, then G is 2-vertex-connected, thus by Menger's Theorem $\exists P_1, P_2$ internally vertex disjoint paths between any $x_i, x_j \in V(G)$.
Thus $x_i P_1 x_j P_2 x_i$ is a cycle.

Let the property hold for, $k = l$. Consider $k = l + 1$.
Let $X \subset V(G)$ be our collection of distinct vertices $\{x_1, \ldots, x_{l+1}\}$.
Let $Y \subset X = \{x_1, \ldots, x_l\}$. Let C be a cycle containing Y.
Since G is k-connected, $\exists\, l + 1$ vertex disjoint paths from x_{l+1} to the collection of endpoints $C_x \subseteq C = \{c_1, \ldots, c_{l+1}\}$.

C_x divides C into $l + 1$ paths, one of which, by PHP, does not contain any $x_i \in Y$. Let c_i, c_j be the endpoints of this path. If $c_i \neq c_j$, attach x_{l+1} to c_i, c_j accordingly and you've formed a cycle containing X.

To avoid $c_i = c_j$ we can pick c_i s.t c_i is the last endpoint we encounter before the first vertex, x_1, when traversing C. Similarly pick c_j s.t c_j is the first endpoint we encounter after the last vertex, x_l. If this process results in $c_i = c_j$, set c_j to x_l which can be done since $\exists$ a path x_{l+1} to x_l that does not include any $x_i \in Y$. If $c_i = c_j$ still, then set $c_i = x_1$, which can be done as by Fan Lemma $\exists$ 2 internally disjoint paths to distinct vertices of $\{c_i, c_j\}$. Obviously $\exists$ a path P between c_i, c_j s.t. $Y \in P$, as before attach x_{l+1} to c_i and c_j accordingly.
□

The converse of the statement is not true as consider the graph of a circle on large N vertices, then $\forall k\ 2 < k \leq N$ a group of k vertices is part of a cycle but this cycle is not k-vertex-connected.
□

6. a. (Fan lemma) Let G be a k-connected graph, let v be a vertex of G, and let X be a set of at most k other vertices. Then there exist $|X|$ internally vertex-disjoint paths from v to distinct vertices of X.

Form G' by adding a vertex q that connects to all of X. $d(q) = |X|$ and by $|X| \leq k$, G' is now $|X|$-connected. By Menger's theorem $\exists\ |X|$ internally vertex-disjoint paths from v to q. Delete q from these paths and you have obtained the desired paths from v to X as all paths from v to q must run through X and cannot have used the same vertex in X.
□

6. b. A graph G is k-linked if, for every choice of $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exist k pairwise vertex-disjoint paths $P_1, \ldots, P_k$ such that each P_i has ends s_i and t_i. Show that there exists a function $f(k)$ such that every $f(k)$-connected graph is k-linked.

Hint: It suffices to take $f(k) = 100k^2$. Pick two large disjoint sets A and B. Use Menger's theorem to find many disjoint A–B paths, use the fan lemma to attach each s_i, t_i to A and B respectively, and then reroute accordingly.

Let $f(k) = 100k^2$.
We will begin picking elements for A and B from $N(s_i)$ and $N(t_i)$.
Note that $|N(s_i)|, |N(t_i)| \geq 100k^2$. Thus $|(\bigcup_i N(s_i)) \cup (\bigcup_i N(t_i))| \geq 100k^2$.
Thus there is plenty of room to start assigning a system of distinct representatives $a_i \in N(s_i)$, $b_i \in N(t_i)$, $a_i, b_i \neq s_i, t_i$.
A and B are now disjoint sets of size k. By Menger's $\exists$ $100k^2$ vertex disjoint paths between A and B. By construction we already have disjoint paths from s_i to a_i and t_i to b_i, so it suffices to find a set of k disjoint paths between each a_i, b_i. Pick one path for each a_i to b_i, call it P_i. Up to $c \leq k$ paths in $\bigcup_i P_i$ may share one or more shared vertices, if so delete $c-1$ of these paths and repeat. This uses up less than $O(k^2)$ paths $<< 100k^2$. Thus $\bigcup_i P_i$ is now pairwise vertex-disjoint and G is k-linked.
□

7. Show that for each $k \in N$, there is some $f(k) \in N$ with the property that every graph G with chromatic number at least $f(k)$ contains a subgraph H with both vertex-connectivity and chromatic number at least k.

Let H be the complete graph on $k+1$ vertices. Let $\chi(G) \geq f(k) \geq k+1$.

Obviously H is k-vertex connected. It takes $(k+1)$-colors to color the k neighbors of v since they are all connected to each other, plus v itself $\forall v \in V(H)$.

Considering $\chi(G)$ is a min value, we know that we cannot remove a color x from the set of colors used in G, $\{C_1 \ldots C_{\chi(G)}\}$, and replace all its colored vertices by another color in the set $\{C_1 \ldots C_{\chi(G)}\} \setminus C_x$. Let S_i contain the vertices of C_i we can recolor. Now $C_i \setminus S_i$ is composed just of vertices that are connected to every other color. Let G' be the induced subgraph of G on $\{\bigcup(C_i \setminus S_i)\}$.

Now consider deleting vertices of G', going through the sets of $C_i \setminus S_i$ one at a time deleting vertices in $C_i \setminus S_i$ and corresponding vertices in $C_j \setminus S_j$ that can now be recolored until $|C_i \setminus S_i| = 1$. This obviously can be done since once you reach $|C_i \setminus S_i| = 1$ you know $\forall\ 1 \leq j \leq \chi(G)$ that $|C_j \setminus S_j| \geq 1$ by the nature of $v \in (C_i \setminus S_i)$ having a neighbor in every other color.

G' is now $K_{\chi(G)}$. By $\chi(G) \geq f(k) \geq k+1$, $H \subseteq G' \subseteq G$.
□