

# Graph Theory Problem Set 1

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1. Show that if  $d_1 \geq d_2 \geq \dots \geq d_n$  is the degree sequence of some graph, then  $\sum_i d_i$  is even and  $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}$  for each  $1 \leq k \leq n$ . Can you also prove that these two conditions are sufficient for  $d_1 \geq d_2 \geq \dots \geq d_n$  to be the degree sequence of some graph?

Condition 1:  $\sum_i d_i$  is even.

Since every edge contributes one degree to two vertices,  $\sum_i d_i = 2e(G)$ , since  $e(G) \in \mathbb{N}$ ,  $\sum_i d_i$  is even.

□

Condition 2:  $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}$  for each  $1 \leq k \leq n$ .

Let  $k = 1$ , then  $d_1 \leq 1(0) + \sum_{i=2}^n \min\{d_i, 1\} = \sum_{i=2}^n \min\{d_i, 1\} = n - 1$  when  $d_i \geq 1 \forall i$ . Conceptually  $d_1 \leq \Delta(G)$  and  $n - 1$  is  $\Delta(G)$ 's tightest upper bound. Let  $S \subseteq V(G)$  s.t.  $S = \{v \in V(G) | d(v) = 0\}$ ,  $S$  is then composed of isolated vertices. Thus we can refine our equation to  $d_1 \leq n - 1 - |S| = \sum_{i=2}^n \min\{d_i, 1\}$  when  $|S| \leq n - 1$  and  $d_1 = 0$  when  $|S| = n$ . Therefore the property holds because  $d_1$  cannot be greater than the amount of all other non-isolated vertices. ✓

Similarly  $\sum_{i=1}^n d_i \leq n(n-1)$ , so holds for  $k = n$ . ✓

Let  $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}$  hold for  $k = x$ ,  $1 \leq x \leq n - 2$ . Consider  $k = x + 1$ .

$$\sum_{i=1}^{x+1} d_i = \sum_{i=1}^x d_i + d_{x+1} \leq x(x-1) + d_{x+1} + \sum_{i=x+1}^n \min\{d_i, x\} = x^2 - x + d_{x+1} + \min\{d_{x+1}, x\} + \sum_{i=x+2}^n \min\{d_i, x\}$$

Case 1:  $d_{x+1} \leq x$ , then

$$\begin{aligned} & x^2 - x + d_{x+1} + \min\{d_{x+1}, x\} + \sum_{i=x+2}^n \min\{d_i, x\} \\ & \leq x^2 - x + x + \min\{d_{x+1}, x\} + \sum_{i=x+2}^n \min\{d_i, x+1\} \\ & = (x+1)((x+1)-1) + \sum_{i=(x+1)+1}^n \min\{d_i, x\} \quad \checkmark \end{aligned}$$

Case 2:  $d_{x+1} > x$ , then

Let  $d_{x+1} = y + 1$  with  $1 \leq y \leq n - 2$ .

We can derive the maximal construction for  $\sum_{i=1}^{x+1} d_i$  by allowing  $x$  vertices to be maximally connected, thus  $\sum_{i=1}^{x+1} d_i \leq x(n-1) + y + 1 = nx - x + y + 1$ .

Recall conditional statement with  $k = x + 1$ :

$$\sum_{i=1}^{x+1} d_i \leq x + 1(x + 1 - 1) + \sum_{i=x+1+1}^n \min\{d_i, x + 1\}$$

Letting  $d_i = y + 1 \forall i \geq x + 2$  (maximal construction)

$$\begin{aligned} &= x^2 + x + \sum_{i=x+2}^n x + 1 \\ &= x^2 + x + (n + 1 - x - 2)(x + 1) \\ &= nx + n - x - 1 \end{aligned}$$

So,  $(nx - x + y + 1 \leq nx + n - x - 1) \rightarrow (y + 2 \leq n) \checkmark$

Thus by induction on the size of  $k$ ,  $1 \leq k \leq n$ ,

$$\sum_{i=1}^k d_i \leq k(k - 1) + \sum_{i=k+1}^n \min\{d_i, k\}$$

□

Claim:

These two conditions are NOT sufficient for  $d_1 \geq d_2 \geq \dots \geq d_n$  to be the degree sequence of some graph.

Proof of Claim:

The first condition says that every edge has  $\{0, 2, 4, 6, 8, \dots\}$  connected vertices.

The second condition says that every collection  $U$  of the first  $k$  vertices is connected to itself  $\sum_{x \in U} d_U(x) \leq k(k - 1)$  times and connected to  $(n - k)$  other vertices outside  $U$   $\sum_{x \notin U} \min\{d_i, k\}$ , where  $\min\{d_i, k\}$  is derived from the fact that no vertex outside  $U$  can connect into  $U$  more times than it's degree, if  $d_i \leq k$ , and no more than  $k$  times if  $k < d_i$ .

Observe the second condition implies edges are unique, that is there cannot be more than one edge that connects two vertices. Otherwise a vertex could connect into a collection of size  $k = 1$  vertices more than once.

However, this is not enough to show that every edge has only two connected vertices.

First, if we imagine an edge with no connected vertices, these conditions are satisfied,  $d_1 \geq d_2 \geq \dots \geq d_n$  is untouched, but this is not a proper graph as  $E(G) \not\subseteq \binom{V(G)}{2}$ .

Second, if we imagine  $K_{2n}$  but allow one edge to connect all the vertices at the same time,  $d(x) = 1 \forall x \in V(G)$ , these conditions are satisfied,  $d_1 \geq d_2 \geq \dots \geq d_n$  holds, but this is not a proper graph as  $E(G) \not\subseteq \binom{V(G)}{2}$ .

□

2. Prove that every tree with maximum degree  $\Delta$  has at least  $\Delta$  leaves.

Let  $v_0 \in V(T)$  be a vertex with maximum degree, thus  $d(v_0) = \Delta = |N(v_0)|$ .

Since  $T$  is acyclic  $E(N(v_0)) = \emptyset$ .

Let  $T_0 \subset T$  be defined as an induced subgraph of  $T$  with  $V(T_0) = N(v_0) \cup v_0$ .

Obviously  $T_0$  is connected, and since  $E(N(v_0)) = \emptyset$ ,  $T_0$  is acyclic, thus  $T_0$  is a tree. Since  $d(v_0) = \Delta$  and  $d(v_i) = 1 \forall v_i \in N(v_0)$ ,  $T_0$  has  $\Delta$  leaves.

Claim:

If  $T'$  is a tree s.t.  $T' \subseteq T$  is an induced subgraph, then  $|leaves(T)| \geq |leaves(T')|$ .

Proof of Claim:

Since  $T$  is a tree,  $T'$  must be obtained by a recursive process of deletion of the leaves of  $T$ . Otherwise, if  $d(v) \geq 2$  then  $v$ 's deletion would create a disconnected graph, by the nature of  $T$  being minimally connected & acyclical.

Any deletion of a leaf can affect at most the degree of one other vertex in  $T$ , since the degree of all leaves is one. Thus the deletion of any leaf can create at most one leaf. Therefore,  $|leaves(T)| \geq |leaves(T')|$ .

Since  $T_0 \subset T$ , by claim  $|leaves(T)| \geq \Delta$ .

□

3. Let  $G$  be a graph such that every vertex has even degree. Prove that  $G$  does not have a *bridge*, i.e., an edge whose deletion increases the number of connected components of  $G$ .

By definition, a graph would not contain a bridge if every connected component was at least 2-connected.

Claim:

If every  $v \in V(G)$  has even degree, every  $v \in V(G)$  is a part of a cycle in  $G$  or an isolated vertex.

Proof of Claim:

Suppose not, then  $\exists v \in V(G)$  that's not in a cycle and instead (Since  $G$  is assumed to be finite) a terminating path. Consider the longest of these paths and let  $x$  be the last vertex in this path with  $d(x) \geq 2$ . Let  $y, z \in N(x)$  s.t.  $y \in path$ .

Case 1:  $y, z$  do not share an edge,  $z \notin path$ , then  $path + z$  is longer.  
Contradiction.  $\perp$

Case 2:  $y, z$  do not share an edge,  $z \in path$ , the  $z_z P_y y x z$  is a cycle.  
Contradiction.  $\perp$

Case 3:  $y, z$  share an edge, then  $yxzy$  is a cycle.

Contradiction.  $\perp$

By claim, every connected component in  $G$  is a cycle or an isolated vertex, every cycle is 2-connected and isolated vertices have no edges, thus  $G$  does not have a bridge.

□

4. What is the maximum possible number of edges in a graph on  $2n$  vertices with exactly one perfect matching?

Let  $M$  be our perfect matching. Notice  $|M| = n$ . Consider any two edges in  $M$ . Let  $x_1, y_1$  be the endpoints of edge 1 &  $x_2, y_2$  be the endpoints of edge 2. There are four possible edges that join these endpoints:  $x_1x_2, x_1y_2, y_1x_2, y_1y_2$ .

If  $x_1x_2$  is allowed to be an edge in  $G$ ,  $y_1y_2$  is disallowed. Otherwise you could replace  $x_1y_1$  &  $x_2y_2$  in  $M$  and get a new perfect matching.

If  $x_1y_2$  is allowed to be an edge in  $G$ , a similar argument disallows  $y_1x_2$ .

The parallels follow from allowing  $y_1y_2$  &  $y_1x_2$ .

Thus, there can be a maximum of two edges allowed between each pair of edges in  $M$ .

Thus,  $e(G) \leq n + 2\binom{n}{2} = n + 2(n/2)(n-1) = n + n^2 - n = n^2$

□

5.a. Show that every graph  $G$  has a bipartition  $(A, B)$  of its vertex set such that  $d_B(x) \geq d(x)/2$  for all  $x \in A$  and  $d_A(x) \geq d(x)/2$  for all  $x \in B$ .

Let  $(A, B)$  be a bipartition of  $V(G)$  that maximizes  $e(A, B)$ .

Let  $x$  be a vertex for which the conditions do not hold.

Case 1: Suppose  $x \in A$ , then  $d_B(x) < d(x)/2$ .

Note  $d_A(x) + d_B(x) = d(x)$ , thus  $d_B(x) < d_A(x)$ .

Let  $A' := A \setminus \{x\}$  &  $B' := B \cup \{x\}$ .

$e(A', B') = e(A, B) + d_A(x) - d_B(x)$  &  $(d_B(x) < d_A(x)) \rightarrow (d_A(x) - d_B(x) > 0)$ , thus  $e(A', B') > e(A, B)$ .

Contradiction.  $\perp$

Case 2: Suppose  $x \in B$ , then  $d_A(x) < d(x)/2$ .

Note  $d_A(x) + d_B(x) = d(x)$ , thus  $d_A(x) < d_B(x)$ .

Let  $B' := B \setminus \{x\}$  &  $A' := A \cup \{x\}$ .

$e(A', B') = e(A, B) + d_B(x) - d_A(x)$  &  $(d_A(x) < d_B(x)) \rightarrow (d_B(x) - d_A(x) > 0)$ , thus  $e(A', B') > e(A, B)$ .

Contradiction.  $\perp$

Thus  $x$  does not exist and  $(A, B)$  satisfies the conditions.

□

5.b. Show that there need not exist any bipartition  $(A, B)$  for which we have  $d_B(x) \geq d(x)/2$  for all  $x \in B$  and  $d_A(x) \geq d(x)/2$  for all  $x \in A$ .

Consider the complete graph on 3 vertices,  $K_3$ .

Claim:

No bipartition  $(A, B)$  can satisfy the conditions, besides the trivial where  $A = \emptyset$  or  $B = \emptyset$ .

Proof of Claim:

Observe a bipartition of the vertex set is not possible without one set  $A, B$  containing only one vertex, thus the degree of that vertex into it's own set is zero while it's degree is 2.

$$0 \geq 2/2 = 1$$

Contradiction.  $\perp$

□

5.c. Finally, show that it is possible to find a bipartition  $(A, B)$  with  $d_B(x) \geq d(x)/2 - 1$  for all  $x \in B$  and  $d_A(x) \geq d(x)/2 - 1$  for all  $x \in A$ .

Substituting  $d_A(x) + d_B(x) = d(x)$  into the inequalities for  $x \in A$ .

$$(d_A(x) \geq (d_A(x) + d_B(x))/2 - 1) \rightarrow (d_A(x) \geq d_B(x) - 2).$$

Similarly for  $x \in B$ ,  $d_B(x) \geq d_A(x) - 2$ .

Let  $(A, B)$  be a bipartition of  $V(G)$  that minimizes  $e(A, B)$ .

Let  $x$  be a vertex for which the conditions do not hold.

Case 1: Suppose  $x \in A$ , then  $d_A(x) < d_B(x) - 2$ . Let  $A' := A \setminus \{x\}$  &  $B' := B \cup \{x\}$ .

$$e(A', B') = e(A, B) + d_A(x) - d_B(x)$$

$$e(A', B') + 2 = e(A, B) + d_A(x) - d_B(x) + 2$$

$$(d_A(x) < d_B(x) - 2) \rightarrow (d_A(x) - d_B(x) + 2 < 0)$$

$$\text{So, } (e(A', B') + 2 < e(A, B)) \rightarrow (e(A', B') < e(A, B)).$$

Contradiction.  $\perp$

Case 2: Suppose  $x \in B$ , then  $d_B(x) < d_A(x) - 2$ . Let  $B' := B \setminus \{x\}$  &  $A' := A \cup \{x\}$ .

$$e(A', B') = e(A, B) + d_B(x) - d_A(x)$$

$$e(A', B') + 2 = e(A, B) + d_B(x) - d_A(x) + 2$$

$$(d_B(x) < d_A(x) - 2) \rightarrow (d_B(x) - d_A(x) + 2 < 0)$$

$$\text{So, } (e(A', B') + 2 < e(A, B)) \rightarrow (e(A', B') < e(A, B)).$$

Contradiction.  $\perp$

Thus  $x$  does not exist and  $(A, B)$  satisfies the conditions.

□

6. For any graph  $G$ , show that there exists a bipartition  $(A, B)$  of its vertex set such that both graphs  $G[A]$  and  $G[B]$  have all their degrees even.

Trivial for  $n \leq 3$ .

Suppose the statement holds for  $n$  vertices. Consider,  $G(V, E)$ , on  $n+1$  vertices. Assume  $\exists v \in V(G)$  with  $d(v)$  odd, otherwise  $A = V(G)$ ,  $B = \emptyset$  and we're done.

Let  $H$  be an induced subgraph of  $G$ , with  $V(H) = V(G) \setminus \{v\}$  and  $E(H) = \{xy \in E(G) \mid x, y \notin N(v) \cup \{v\}\} \cup \{ij \mid ij \notin E(G) \text{ \& } i \neq j \in N(v)\}$  i.e  $H$  is the graph of  $G$  with  $v$  deleted and the neighbor's of  $v$  joined to other neighbors if they did not share an edge in  $G$  and disjointed if they did. By induction  $H$  has a bipartition  $(A, B)$  of its vertex set such that both graphs  $H[A]$  and  $H[B]$  have all their degrees even.

Note  $\forall x \notin N(v) \cup \{v\}$ , their edges remain the same in  $G$  &  $H$ , INCLUDING the edges into  $N(v)$ .

$d(v) = d_A(v) + d_B(v)$  and since  $d(v)$  is odd, either  $d_A(v)$  or  $d_B(v)$  is even. WLOG, assume  $d_A(v)$  is even and consider the addition of  $v$  to  $A$ .

Claim:

$$\forall x \in A \cap N(v), d_{G[A]}(x) = d_{H[A \setminus N(v)]}(x) + |A \cap N(v)| - 1 - d_{H[A \cap N(v)]}(x) + 1$$

Proof of Claim:

To find the parity of the degree  $x \in A \cap N(v)$  in  $G$ , it helps to break  $G$  into three parts: 1. The overlap of  $H$  in  $G$ , 2.  $N(v)$  & 3.  $v$ .

The overlap of  $H$  in  $G$  is represented by  $d_{H[A \setminus N(v)]}(x)$  since  $H$  and  $G$  share all edges outside and leading into  $N(v) \cup \{v\}$ , which also makes this a valid partition of  $G$ .

Since  $E(G[N(v)])$  and  $E(H[N(v)])$  are compliments,  $d_{G[A \cap N(v)]}(x) = |A \cap N(v)| - 1 - d_{H[A \cap N(v)]}(x)$ .  $v$  obviously contributes 1 degree.

Finding the parity  $\forall x \in A \cap N(v)$ :

Observe  $d_{H[A \setminus N(v)]}(x) + d_{H[A \cap N(v)]}(x)$  is even, thus they share the same parity.

$$\text{Parity}(d_{H[A \setminus N(v)]}(x)) = \text{Parity}(d_{H[A \cap N(v)]}(x))$$

$\text{Parity}(|A \cap N(v)|)$  assumed even.

$$\text{Parity}(d_{G[A]}(x)) = \text{Parity}(d_{H[A \setminus N(v)]}(x)) + \text{Parity}(|A \cap N(v)|) - \text{Parity}(1) -$$

$$\text{Parity}(d_{H[A \cap N(v)]}(x)) + \text{Parity}(1) \rightarrow \text{Parity}(d_{G[A]}(x)) = \text{Parity}(|A \cap N(v)|)$$

Thus  $\text{Parity}(d_{G[A]}(x))$  is even  $\forall x \in A \cap N(v)$ .

A parallel calculation follows  $\forall x \in B \cap N(v)$ :

Note  $v$  does not contribute any degrees to  $x \in B \cap N(v)$ .

$$\text{Following parallel steps get: } \text{Parity}(d_{G[B]}(x)) = \text{Parity}(|B \cap N(v)|) - \text{Parity}(1)$$

$\text{Parity}(|B \cap N(v)|)$  is odd, thus  $\text{Parity}(d_{G[B]}(x))$  is even  $\forall x \in B \cap N(v)$ .

Wrapping up:

All vertices in  $H[B \setminus N(v)]$  have even degree.

All vertices in  $G[B \cap N(v)]$  have even degree.

$v \notin B$

By claim,  $G[B]$  has all its degrees even. ✓

All vertices in  $H[A \setminus N(v)]$  have even degree.

All vertices in  $G[A \cap N(v)]$  have even degree.

$d_{G[A]}(v)$  is even.

By claim,  $G[A]$  has all its degrees even. ✓

Thus  $(A, B)$  satisfies the claim with  $v$  added to  $A$ .

□